SPATIAL AND TEMPORAL DEVELOPMENT OF DISTURBANCES IN A PLANE CHANNEL

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The relationship between temporal and spatial development of disturbances in hydrodynamic stability is investigated.

It is shown on the basis of a plane Poiseuille flow that the results of numerical calculations by temporal and spatial methods are nearly the same. Approximate transitional formulas are presented.

Two methods may be used to investigate fluid-flow stability.

Temporal Method. At the initial instant of time, perturbations periodic in the spatial coordinate, whose amplitude is significantly lower than the average flow velocity, are imposed on the flow. The rate of change of the perturbation amplitude with time will also characterize the degree of flow stability. From a mathematical point of view this method reduces [1, 2] to the solution of the problem for the eigenvalues of the Orr–Sommerfeld equations

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha R\,[(C-u)(\varphi'' - \alpha^2\varphi) + u''\varphi] \tag{1}$$

$$\Psi = \varphi(y)\, e^{i\alpha(x-Ct)}, \qquad C = X + iY \tag{2}$$

Here $\varphi = \varphi(y)$ is the complex amplitude of the stream function Ψ for perturbations; x is the coordinate along the flow; y is the coordinate perpendicular to the flow; α is a real parameter – the wave number of the perturbations; and C is the desired eigenvalue. The problem is considered for uniform boundary conditions for the function φ.

Spatial Method. At the initial cross section of the channel, small-amplitude perturbations which are periodic with time are imposed on the flow, and their propagation along the flow is studied. In this case the degree of flow stability is characterized by the damping rate of the perturbation amplitude with respect to the spatial coordinate. This method is more "physical" since it corresponds more closely to the conditions of the experiment. It reduces [3] to the problem for the eigenvalues of the equations

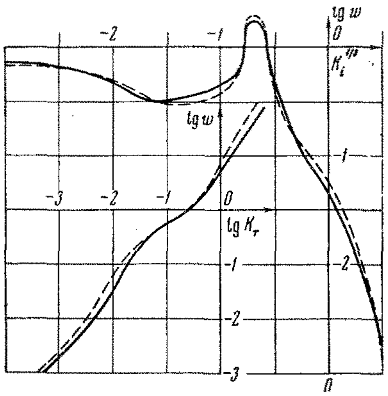

Fig. 1

$$\varphi^{IV} - 2K^2\varphi'' + K^4\varphi = iR\,[(\omega - Ku)(\varphi'' - K^2\varphi) + Ku''\varphi] \tag{3}$$

$$\Psi = \varphi(y)\, e^{i(\omega t - Kx)}, \qquad K = K_r + iK_i \tag{4}$$

Here $\varphi = \varphi(y)$ is the complex amplitude of the stream function ψ for the perturbations; ω is a real parameter – the frequency of oscillations at a given cross section; and K is the desired eigenvalue.

It is of interest to determine whether a qualitative analogy exists between the results obtained by these two methods, since conclusions pertaining to the degree of stability based on only one of them cannot be treated as completely general. To this end we consider both types of perturbations with identical wavelength in the spatial coordinate

$$\alpha = K_r \tag{5}$$

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In the temporal case the change in amplitude with time at the point moving with the phase velocity follows the relationship

$$\varphi(t) = \varphi(t_0)\, e^{\alpha Y t}.$$

In the spatial case for this point $x = (\omega/K_r)t$, and from Eq. (4) we obtain

$$\varphi(t) = \varphi(t_0)\, e^{(K_i\omega/K_r)\, t}.$$

We now follow the change in amplitude at this point over the spatial coordinate. In the spatial case, in accordance with Eq. (4)

$$\varphi(x) = \varphi(x_0)\, e^{K_i x}.$$

In the temporal case, since $x = Xt$ at the given point, from Eq. (2) we obtain

$$\varphi(x) = \varphi(x_0)\, e^{(\alpha Y/X)\, x}.$$

If it is assumed that for equal spatial wavelengths (5) spatial as well as temporal damping coefficients are approximately the same for both cases, we have

$$\alpha Y \approx K_i\omega / K_r, \quad K_i \approx \alpha Y / X, \quad \text{or} \quad X \approx \omega K_r. \tag{6}$$

From these equations we have

$$K_r \approx \alpha, \quad \omega \approx \alpha X. \tag{7}$$

It is shown in [4] that for small values of αY these equalities are valid to an accuracy to $O((\alpha Y)^2)$ in the vicinity of the neutral curve. Naturally, on the neutral curve where $K_i = Y = 0$, there is agreement; at other points such agreement may not occur. In the first case waves of identical amplitude propagate along the flow and are damped or amplified; in the second case waves of different amplitude are involved.

A numerical solution of Eqs. (1) and (3) for a plane Poiseuille flow was obtained by the method described in [2] at a Reynolds number, based on a maximum velocity, of $R = 10^4$ in order to compare the two methods. The solid lines in Fig. 1 denote the results computed by the spatial method, and the dashed lines correspond to the temporal method taking Eqs. (6) and (7) into account. The points corresponding to neutral frequencies coincide, as expected. The deviation at remaining points is not large.

The asymptotic behavior of K_i for small ω may be obtained from Eq. (3) by setting $\omega = K_r = 0$ (since the phase velocity is finite at low frequencies) and $u = 1$ (since the damping decrement for long wavelengths is weakly dependent on the velocity profile). For symmetric functions $\varphi(y)$ this leads to the following equation:

$$\sqrt{K_i^2 - RK_i}\, \operatorname{tg} \sqrt{K_i^2 - RK_i} = K_i \operatorname{tg} K_i$$

whose solution for small K_i is close to the value $K_i = -\pi^2/R$. For temporal damping, in accordance with [5], one may obtain the following asymptotic relationship for small α:

$$\frac{\alpha Y}{X} = -\frac{\pi^2}{RX}, \qquad X \approx 0.62$$

The approximate agreement may be used for the solution of technical problems and also in the choice of starting points for the numerical computation.

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